W15/1 Completions of Rings and Modules

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<u>Cor 9.5</u> Let G be an ab. top. group w. linear topology induced by a filtration $(G_n)_{n\geq 0}$.

(1) If $0 \to H \hookrightarrow G \xrightarrow{\pi} K \to 0$ is a SES of top. groups, with topology on H, K induced from G (on H: $(H \cap G_n)_{n\geq 0}$, on K: $(\pi G_n)_{n \geq 0}$), then

$$0 \to \hat{H} \to \hat{G} \to \hat{K} \to 0$$

is exact.

(2) $\hat{G}_n \leq \hat{G}$ and $\hat{G}/\hat{G}_n \cong G/G_n$.

(3) $\hat{\hat{G}} \cong \hat{G}$. ["The completion is complete."]

<u>Proof:</u> (1) Apply T9.2 to $0 \to H/H\cap G_n \to G/G_n \to K/\pi G_n \to 0$

(2) Apply (1) to $0 \to G_n \to G \to G/G_n \to 0$, to get

$\widehat{(G/G_n)} \cong \hat{G}/\hat{G}_n$. Since G/G_n is discrete, $\widehat{(G/G_n)} \cong G/G_n$.

(3) $\hat{G} \cong \varprojlim_n G/G_n \underset{(2)}{\cong} \varprojlim_n \hat{G}/\hat{G}_n \cong \hat{\hat{G}}$. $\square$

Let A be a ring.

<u>Def:</u> If $I \trianglelefteq A$, and $M \in A\text{-Mod}$, the **I-adic topology** on M is the one induced by $M \supseteq IM \supseteq I^2M \supseteq \dots$.

$\hat{M} = \varprojlim_n M/I^nM$ is the **I-adic completion** of M.

<u>Properties (w/o proof)</u>

1) A is a topological ring (operations are continuous), and so is $\hat{A}$.

$\ker(A \to \hat{A}) = \bigcap_{n\geq 0} I^n$, A Hausdorff $\Leftrightarrow \bigcap_{n \geq 0} I^n = 0$.

2) $\hat{M}$ is a top. $\hat{A}$-mod. ($\hat{A} \times \hat{M} \to \hat{M}$ is continuous)

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3) If $f\in\mathrm{Hom}(M,N)$ with $M,N\in A\text{-Mod}$, then f is continuous [follows from $f(I^nM)=I^nf(M)\subseteq I^nN$.], so induces a cont. hom. $\hat{f}:\hat{M}\to\hat{N}$.

<u>Def:</u> $I\trianglelefteq A$, $M\in A\text{-Mod}$, $M=M_0\supseteq M_1\supseteq\ldots$ filtration.

(1) $(M_n)_{n\geq0}$ is an I-filtration if $IM_n\subseteq M_{n+1}$ $\forall n\geq0$

(2) $(M_n)_{n\geq0}$ is stable if moreover $I\cdot M_n=M_{n+1}$ for all suff. large n.

<u>Lemma 9.6</u> Stable I-filtrations $(M_n)_{n\geq0}$, $(M_n')_{n\geq0}$ of M have bounded difference, i.e.,

$$\exists c\geq0\ \forall n\geq0:\quad M_{n+c}\subseteq M_n' \text{ and } M_{n+c}'\subseteq M_n.$$

In particular: they determine the same topology on M (the I-adic one)

<u>Proof:</u> Suffices to consider $M_n'=I^nM$.

$IM_n\subseteq M_{n+1}$ and $M=M_0$ $\Rightarrow$ $I^nM\subseteq M_n$.

Let $n_0\geq0$ s.t. $\forall n\geq n_0$: $IM_n=M_{n+1}$ $\Rightarrow$ $M_{n+n_0}=I^nM_{n_0}\subseteq I^nM$ $\square$

<u>Def:</u> Let $I\trianglelefteq A$, $M\in A\text{-Mod}$, with I-filtration $(M_n)_{n\geq0}$.

$$R(I,A):=A[It]=\bigoplus_{n=0}^{\infty}I^n\underset{\text{formal}}{t^n}=A\oplus It\oplus I^2t^2\oplus\ldots$$ is

the Rees algebra of A w.r.t. I, and

$$R(I,M):=\bigoplus_{n=0}^{\infty}M_nt^n=M\oplus M_1t\oplus M_2t^2\oplus\ldots$$ is a Rees module

<u>Note:</u> $R(I,A)$ is an A-algebra:

$\forall x\in I^m, y\in I^n$: $(xt^m)(yt^n)=xyt^{m+n}$, and

$R(I,M)$ is a $R(I,A)$-module:

$\forall x\in I^m\ \forall y\in M_n$: $(xt^m)(yt^n)=\underbrace{xy}_{\in M_{m+n}}t^{m+n}$. bec. I-filtration

$\forall x \in I^m\ \forall y \in M_n: \quad (xt^m)(yt^n) = \underbrace{xy}_{\in M_{m+n}} t^{m+n}$. bec. I-filtration

<u>Prop 9.10</u> If A is noetherian, $M \in A$-Mod f.g. with I-filtration $(M_n)_{n\geq 0}$, then

$$(M_n)_{n\geq 0} \text{ stable} \iff R(I,M) \text{ is f.g. as } R(I,A)\text{-module.}$$

<u>Proof</u>: $I\cdot t$ generates $R(I,A)$ as A-algebra. A noetherian $\Rightarrow I$ f.g.
$\xrightarrow[\text{theorem}]{\text{Hilbert basis}}$ $R(I,A)$ noetherian.

The $R(I,A)$-submodule of $R(I,M)$ generated by $\bigoplus_{n=0}^{k} M_n t^n$ is

$$U_k := \bigoplus_{n=0}^{k} M_n t^n \oplus \bigoplus_{n=k+1}^{\infty} I^{n-k} M_k t^n \qquad (k\geq 0)$$

and $R(I,M) = \bigcup_{k\geq 0} U_k$. Note that U_k is a f.g. $R(I,A)$-module because each M_n is a f.g. A-module.

$R(I,M)$ is a f.g. $R(I,A)$-module $\iff \exists k\geq 0: \ R(I,M) = U_k$

$\iff U_0 \subseteq U_1 \subseteq U_2 \ldots$ stabilizes

$\iff \exists k\geq 0 : \forall n\geq k: \ I^{n-k} M_k = M_n$

$\iff (M_n)_{n\geq 0}$ is stable $\square$

<u>Thm 9.11</u> Let A be noetherian, $I \trianglelefteq A$, M f.g. A-module, $M' \leq M$.

(1) If $(M_n)_{n\geq 0}$ is a stable I-filtration of M, then $(M_n \cap M')_{n\geq 0}$ is a stable I-filtration of M'.

(2) $(I^n M')_{n\geq 0}$ and $(I^n M \cap M')_{n\geq 0}$ have bounded differences. In particular: The I-adic top. on M' coincides with the subspace top. induced from the I-adic top. on M.

(3) [Artin-Rees Lemma]

$$\exists k\geq 0\ \forall n\geq k: \quad (I^n M) \cap M' = I^{n-k}((I^k M)\cap M')$$

$$\exists k \geq 0\ \forall n \geq k: \quad (I^n M) \cap M' = I^{n-k}\big((I^k M) \cap M'\big)$$

Proof: (1) $I(M_n \cap M') \subseteq IM_n \cap IM' \subseteq M_{n+1} \cap M'$, so $(M_n \cap M')_{n\geq 0}$ is an I-filtration. $R(I,M)$ is a noetherian $R(I,A)$-module (P 9.10), and $R(I,M')$ is a submodule, so f.g. Thus $(M_n \cap M')_{n\geq 0}$ is stable (P9.10).

(2) By L.9.6.

(3) Apply (1) to $(I^n M)_{n\geq 0}$. $\square$

Cor 9.12 A noetherian, $I \trianglelefteq A$, M, N, K f.g. A-modules.
If $0 \to M \to N \to K \to 0$ is exact, so is $0 \to \hat{M} \to \hat{N} \to \hat{K} \to 0$
(I-adic topology)

Proof: C9.5(1) + T9.11. $\square$

Lemma 9.13 (Nakayama, explicit form) If $M \in A\text{-Mod}$ is f.g. and $IM = M$, there exists $x \in A$ s.t. $xM = 0$ and $x \equiv 1 \mod I$.

Proof: Localize A at $S = 1 - I \overset{1.1.1(2)}{\Longrightarrow} S^{-1}I \subseteq \mathfrak{J}(S^{-1}A)$

[Show: $\forall x \in I, a \in A, s \in S$: $1 - \frac{a}{s}x \in (S^{-1}A)^\times$.
Let $s = 1 - \underbrace{y}_{\in I}$ with $y \in I$. $1 - \frac{a}{s}x \in (S^{-1}A)^\times \Leftrightarrow s - ax \in (S^{-1}A)^\times$
$\Leftrightarrow 1 - \overbrace{y - ax}^{\in I} \in (S^{-1}A)^\times$, true since $1 - y - ax \in S$.]

$$(S^{-1}I)(S^{-1}M) = S^{-1}M \underset{L1.5}{\overset{\text{Nakayama}}{\Longrightarrow}} S^{-1}M = 0 \overset{M \text{ f.g.}}{\Longrightarrow} \exists x \in S: \ xM = 0. \quad \square$$

Thm 9.14 A noeth, $I \trianglelefteq A$, M f.g. A-module w. I-adic topology.
Then $\operatorname{Ker}(M \to \hat{M}) = \{m \in M : \exists x \in I: (1-x)m = 0\}$

Proof: $K := \operatorname{Ker}(M \to \hat{M}) = \bigcap_{n \geq 0} I^n M$.

„$\subseteq$": K is the intersection of all nbhds of 0 (L9.3), so it has the trivial topology in the subspace topology.
Since $\emptyset \neq \underset{\underset{0}{\Downarrow}}{IK} \subseteq K$ is open, $IK = K \overset{L9.13}{\Longrightarrow} \exists x \in I: \ (1-x)m = 0$.

„$\supseteq$": Suppose $x \in I$, $m \in M$ with $(1-x)m = 0 \Rightarrow m = xm \Rightarrow m = x^n m\ \forall n$

Since $0 \in IK \subseteq K$ is open, $IK = K \Rightarrow \exists x \in I: \ (1-x)m = 0$.

"$\supseteq$": Suppose $x \in I$, $m \in M$ s.t. $(1-x)m = 0 \Rightarrow m = xm \Rightarrow m = x^n m \ \forall n \geq 0$

$\Rightarrow m \in \bigcap_{n\geq 0} I^n M = K$. $\square$

<u>Cor 9.15</u> If A is noetherian, $I \trianglelefteq A$, $I \subseteq \mathfrak{J}(A)$, M f.g. A-module, then $\bigcap_{n\geq 0} I^n M = \underline{0}$, i.e., M is Hausdorff in the I-adic topology.

<u>Proof</u>: $\forall x \in I$: $1 - x \in A^\times$, so T9.14 shows $\operatorname{Ker}(M \to \hat{M}) = \underline{0}$. $\square$

<u>Cor 9.16</u> (Krull's Intersection Theorem) If A is noetherian and ($I \subseteq \mathfrak{J}(A)$ <u>OR</u> A is a domain and $I \neq A$), then $\bigcap_{n\geq 0} I^n = \underline{0}$.

<u>Proof</u>: 1st Case by C9.15, $M = A$. Suppose A is a domain, $M \in \operatorname{Max}(A)$ s.t. $I \subseteq M \Rightarrow A_M$ is a local noetherian ring $\Rightarrow \bigcap_{n\geq 0} I^n A_M = 0$.

Since $I^n \subseteq I^n A_M$, also $\bigcap_{n\geq 0} I^n = \underline{0}$. $\square$.

<u>Lemma 9.17</u> A noetherian, $I \trianglelefteq A$.

(1) $\forall n \geq 0$: $I^n \hat{A} = \widehat{I^n}$

(2) $\forall n \geq m \geq 0$: $I^m / I^n \cong \widehat{I^m} / \widehat{I^n}$

(3) $\hat{I} \subseteq \mathfrak{J}(\hat{A})$

<u>Proof</u>: (1) By C9.12, there is a comm. diagram

$$\begin{tikzcd} 0 \arrow[r] & I^n \arrow[r, hook] \arrow[d, "\varphi|_{I^n}"] & A \arrow[r] \arrow[d, "\varphi"] & A/I^n \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \widehat{I^n} \arrow[r, hook] & \hat{A} \arrow[r] & A/I^n \arrow[r] & 0 \end{tikzcd}$$

where $\widehat{A/I^n} = A/I^n$, bec. A/I^n is discrete in the I-adic topology.

Diagram chase: $\varphi(I^n) = \hat{I}^n$.

(2) Consider

$$\begin{tikzcd} 0 \arrow[r] & I^n \arrow[r] \arrow[d] & I^m \arrow[r] \arrow[d] & I^m/I^n \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & \widehat{I^n} \arrow[r] & \widehat{I^m} \arrow[r] & I^m/I^n \arrow[r] & 0 \end{tikzcd}$$

(again, I^m/I^n is discrete)

(3) Let $x \in \hat{I}$. Suffices to show: $1 - x \in \hat{A}^\times$ (L.A.1)

$\longrightarrow I \longrightarrow /I^n \to 0$

(3) Let $x \in \hat{I}$. <u>Suff. to show:</u> $1 - x \in \hat{A}^\times$ (L1.1).

For $k \geq 0$, let $s_k := \sum_{n=0}^{k} x^k$. If $k, h \geq N$, then $s_k - s_h \in \hat{I}^N$,

so $(s_k)_{k\geq 0}$ defines an element s of $\varprojlim_k \hat{A}^k/\hat{I}^k \underset{(2)}{\cong} \varprojlim_k A^k/I^k = \hat{A}$

Now $(1-x)s = 1$ in $\hat{A}$ (consider mod $\hat{I}^N$ for $N \to \infty$).

<u>Thm 9.18</u> If (A, M) is a local noetherian ring, then $\hat{A}$ is local with maximal ideal $\hat{M}$ (Completion w.r.t. M-adic topology)

<u>Proof:</u> Since $A/M \cong \hat{A}/\hat{M}$ (L9.17(2)), $\hat{A}/\hat{M}$ is a field, so $\hat{M}$ is maximal. Since $\hat{M} \subseteq J(\hat{A})$ (L9.17(3)), in fact $\hat{M} = J(\hat{A})$. $\square$

<u>Thm 9.19</u> If A is noetherian, $I \trianglelefteq A$, then the I-adic completion $\hat{A}$ is also noetherian

<u>Proof Idea:</u> Either using associated graded rings [AM69, Thm. 10.26] or directly [AF23, Thm 7.5.22]:

<u>First:</u> A noetherian $\Rightarrow A[[x_1, \dots, x_n]]$ noetherian

[Induction on n, since $A[[x_1, \dots, x_n]] \cong A[[x_1, \dots, x_{n-1}]][[x_n]]$.

Like Hilbert's Basis Theorem, but work with the lowest coefficient.

<u>Second:</u> Let $I = \langle a_1, \dots, a_n \rangle_A$ Show there exists a surjective ring. hom. $\varphi: K[[x_1, \dots, x_n]] \longrightarrow \hat{A} = \varprojlim_m A/I^m$. $\square\approx$

<u>Exm:</u> (1) $\mathbb{Z} \hookrightarrow \mathbb{Z}_{(p)} = \{\frac{a}{b} : p \nmid b\} \hookrightarrow \mathbb{Z}_p = \{\sum_{i=0}^{\infty} a_i p^i : a_i \in \{0, \dots, p-1\}\}$

(2) X variety, $\underline{a} \in X$, $I(\underline{a})$ max. ideal of $\underline{a}$,

$$A(X) \longrightarrow A(X)_{I(\underline{a})} = \left\{\frac{f}{g} : g(\underline{a}) \neq 0\right\} \xhookrightarrow[\text{Krull's intersection theorem}]{} \widehat{A(X)}_{I(\underline{a})}$$

Krull's intersection theorem

If $I(\underline{a})$ is regular, $\widehat{A(X)_{I(a)}} \cong K[\![x_1, \dots, x_d]\!]$, $d = \dim(A(X)_{I(\underline{a})})$,

by Cohen's Structure Theorem (not proven; [AF23, Cor. 10.6.7])